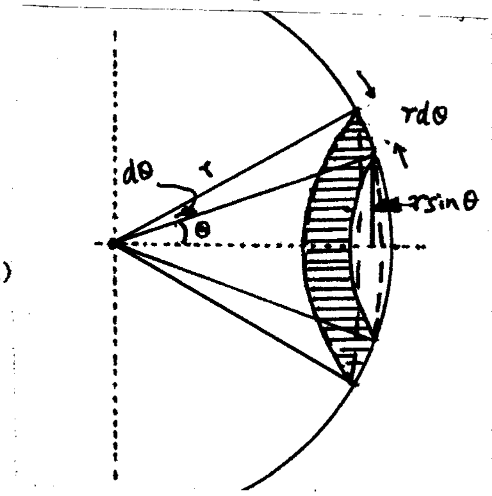
KINETIC THEORY DERIVATIONS

Preliminary proof in geometry

We shall be considering the arrival of a particle at an orifice or wall at some specified angle. Because there is cylindrical symmetry, all approaches along an imaginary cone of half-angle θ are equivalent. We are thus interested in the fraction, F, of all particles that lie within cones of half angles θ and $\theta+d\theta$. Consider the two cones bounded by a sphere of radius r. F is then the ratio of the volume between the two cones to the total volume of the sphere. Referring to the figure, it is evident that this ratio is also the ratio of the shaded area, dA, to the area of the whole sphere ($4\pi r^2$). The radius of the shaded ring is $r\sin\theta$, and its thickness is $rd\theta$, so that its area

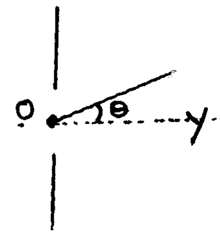


$$dA = 2\pi . r\sin\theta . rd\theta \qquad (1)$$

and

$$F = dA/A = 2\pi . r\sin\theta . rd\theta / 4\pi r^2 = (1/2)\sin\theta . d\theta$$

Flux of molecules and effusion



If n is the number of particles per unit volume, then the number between the two cones and with velocities between c and c+dc

$$= n.F.f(c)dc$$

Thus number crossing area A in time dt, and at an angle θ to OY

$$= n.F.f(c)dc.A.c\cos\theta.dt$$

Substitution of expression (1) for F and integration from $\theta = 0$ to $\theta = \pi/2$ and from $c = 0$ to $c = \infty$ gives total number crossing area per unit time

$$\frac{dn}{dt} = \frac{1}{2} n.A. \int_0^{\pi/2} \sin\theta\cos\theta d\theta . \underbrace{\int_0^{\infty} cf(c)dc}_{\bar{c} \text{ by definition}}$$

$$\boxed{= \frac{1}{4} n.A.\bar{c}}$$

Pressure

Derivation follows the previous line, but with a calculation of the momentum transferred within the given ranges of direction and velocity.

$$\left(\frac{dp}{dt}\right)_{\substack{\theta\to\theta+d\theta \\ c\to c+dc}} = n.F.f(c)dc.A.c\cos\theta.2mc\cos\theta$$

Thus

$$\text{Force} = \left(\frac{dp}{dt}\right)_{\text{TOTAL}} = m.n.A \int_0^{\pi/2} \cos^2\theta\sin\theta d\theta \underbrace{\int_0^{\infty} c^2 f(c)dc}_{\overline{c^2} \text{ by definition}}$$

so that

$$P = \frac{\text{Force}}{A} = \boxed{\frac{1}{3} n.m.\overline{c^2}}$$